

ON (GORENSTEIN) INJECTIVE AND (GORENSTEIN) FLAT DIMENSIONS OF RIGHT DERIVED SECTION FUNCTOR OF COMPLEXES

CYRUS JALALI

ABSTRACT. Let (R, \mathfrak{m}) be a commutative Noetherian local ring, \mathfrak{a} be a proper ideal of R and M be an R -complex in $D(R)$. We prove that if $M \in D_{\square}^f(R)$ (respectively, $M \in D_{\square}^f(R)$), then $\mathrm{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{id}_R M$ (respectively, $\mathrm{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{fd}_R M$). Next, it is proved that the right derived section functor of a complex $M \in D_{\square}(R)$ (R is not necessarily local) can be computed via a genuine left-bounded complex $G \simeq M$ of Gorenstein injective modules. We show that if R has a dualizing complex and M is an R -complex in $D_{\square}^f(R)$, then $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$ and $\mathrm{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gid}_R M$. Also, we show that if M is a relative Cohen-Macaulay R -module with respect to \mathfrak{a} (respectively, Cohen-Macaulay R -module of dimension n), then $\mathrm{Gfd}_R \mathbf{H}_{\mathfrak{a}}^{\mathrm{ht}_M \mathfrak{a}}(M) = \mathrm{Gfd}_R M + n$ (respectively, $\mathrm{Gid}_R \mathbf{H}_{\mathfrak{m}}^n(M) = \mathrm{Gid}_R M - n$). The above results generalize some known results and provide characterizations of Gorenstein rings.

1. INTRODUCTION

Throughout this paper, R is a commutative Noetherian ring, \mathfrak{a} is a proper ideal of R and M is an R -complex. The category of R -complexes is denoted $C(R)$, and we use subscripts \square , \square and \square to denote genuine boundedness conditions. Also, the derived category is denoted $D(R)$, and we use subscripts \square , \square and \square to denote homological boundedness conditions. The symbol \simeq is the sign for isomorphism in $D(R)$ and quasi-isomorphisms in $C(R)$. We also use superscript f to signify that the homology modules are degreewise finitely generated. An R -complex I is semiinjective if the functor $\mathrm{Hom}_R(-, I)$ converts injective quasiisomorphisms into surjective quasiisomorphisms. A semiinjective resolution of an R -complex M is a semiinjective complex I and a quasiisomorphism $M \xrightarrow{\sim} I$. For an R -complex M the injective dimension $\mathrm{id}_R M$ is defined as

$$\mathrm{id}_R M = \inf \left\{ \ell \in \mathbb{Z} \mid \begin{array}{l} \exists \text{ semiinjective } R\text{-complex } I \text{ such that} \\ M \simeq I \text{ in } D(R) \text{ and } I_v = 0 \text{ for all } v < -\ell \end{array} \right\}.$$

Several of the main results of this paper involve the hypothesis that R has a dualizing complex. A complex $D \in D_{\square}^f(R)$ is dualizing for R if it has finite injective dimension and the canonical morphism $\chi_M^R : R \rightarrow \mathbf{R}\mathrm{Hom}_R(D, D)$ is an isomorphism in $D(R)$. If R has a dualizing complex D , we may consider the functor $-^{\dagger} = \mathbf{R}\mathrm{Hom}_R(-, D)$. The notion of Gorenstein injective module was introduced by E.E. Enochs and O.M.G. Jenda in [6]. An R -module M is said to be Gorenstein injective, if there exists a $\mathrm{Hom}_R(\mathcal{I}, -)$ exact acyclic complex E of injective R -modules such that $M = \mathrm{Ker}(E_0 \rightarrow E_{-1})$. The Gorenstein injective dimension, $\mathrm{Gid}_R M$, of $M \in D_{\square}(R)$ is defined to be the infimum of the set of integers n such that there exists a complex $G \in C_{\square}(R)$ consisting of Gorenstein injective modules satisfying $M \simeq G$ and $G_n = 0$ for $n < -\ell$. Also, an

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R -complex F is semiflat if the functor $- \otimes_R F$ preserves injective quasiisomorphisms. For an R -complex M the flat dimension $\text{fd}_R M$ is defined as

$$\text{fd}_R M = \inf \left\{ n \in \mathbb{Z} \mid \begin{array}{l} \exists \text{ semiflat } R\text{-complex } F \text{ such that} \\ F \simeq M \text{ in } D(R) \text{ and } F_v = 0 \text{ for all } v > n \end{array} \right\}.$$

An R -module M is said to be Gorenstein flat, if there exists an $\mathcal{I} \otimes_R$ -exact acyclic complex F of flat R -modules such that $M = \text{Ker}(F_0 \rightarrow F_{-1})$. The Gorenstein flat dimension, $\text{Gfd}_R M$, of $M \in D_{\square}(R)$ is defined to be the infimum of the set of integers n such that there exists a complex $F \in C_{\square}(R)$ consisting of Gorenstein flat modules satisfying $M \simeq F$ and $G_n = 0$ for $n < \ell$. Let M be an R -complex in $D(R)$. The right derived section functor of the complex M is defined as $\mathbf{R}\Gamma_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(E)$, where E is a semiinjective resolution of M (see [8] and [16]).

It has been shown in [7, Theorem 6.5] that the right derived section functor (with support in any ideal \mathfrak{a}) sends complexes of finite flat dimension (respectively, finite injective dimension) to complexes of finite flat dimension (respectively, finite injective dimension). In section 2, as a main result, we prove that if (R, \mathfrak{m}) is a local ring and $M \in D_{\square}^f(R)$, then $\text{id}_R M$ and $\text{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)$ are simultaneously finite, and that $\text{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{id}_R M$ (see Theorem 2.6). Then, as a corollary, we provide a characterization of Gorenstein rings which recovers [20, Corollary 2.7]. Also, in 2.10, we prove that if (R, \mathfrak{m}) is a local ring and M is an R -complex in $D_{\square}^f(R)$ with $\sup \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$, then $\text{id}_R M$ and $\text{id}_R \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ ($\mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ is considered as a module) are simultaneously finite and there is an equality $\text{id}_R \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \text{id}_R M - n$. Notice that, this result is a generalization of [20, Theorem 2.5].

In section 3, as a main result, a Gorenstein flat version of 2.6 is demonstrated. Indeed, it is shown, in Theorem 3.5, that if (R, \mathfrak{m}) is a local ring and $M \in D_{\square}^f(R)$, then $\text{fd}_R M$ and $\text{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)$ are simultaneously finite, and that $\text{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{fd}_R M$. Next, as an application of Theorem 3.5, we prove a Gorenstein flat version of 2.10 (see Theorem 3.7). It has been proved in [4, Theorem 5.9] that if $M \in D_{\square}(R)$, then

$$\text{Gid}_R M < \infty \Rightarrow \text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty.$$

Moreover, if R has a dualizing complex, the above implication may be reversed if \mathfrak{a} is in the Jacobson radical of R and $M \in D_{\square}^f(R)$. In Theorem 3.10, as a main result, we show that if (R, \mathfrak{m}) is a local ring admitting a dualizing complex and M is an R -complex in $D_{\square}^f(R)$, then $\text{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{Gfd}_R M$. Then, as a corollary, we prove that if M is a relative Cohen-Macaulay R -module with respect to \mathfrak{a} , where is defined as in [20], and that $n = \text{grade}(\mathfrak{a}, M)$, then $\text{Gfd}_R \mathbf{H}_{\mathfrak{a}}^n(M) = \text{Gfd}_R M + n$.

In section 4, we prove that if C is an $\Gamma_{\mathfrak{a}}$ -acyclic R -complex in $C_{\square}(R)$, then $\mathbf{R}\Gamma_{\mathfrak{a}}(C) \simeq \Gamma_{\mathfrak{a}}(C)$ (see Theorem 4.5). As an application of this Theorem, in Corollary 4.7, we show that, the right derived section functor of a complex $M \in D_{\square}(R)$ can be computed via a genuine left-bounded complex $G \simeq M$ of Gorenstein injective modules. Also, as a main result, we show that if (R, \mathfrak{m}) is a local ring admitting a dualizing complex and M is an R -complex in $D_{\square}^f(R)$, then $\text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{Gid}_R M$ (see Theorem 4.10). As a corollary, we provide a characterization of Gorenstein rings which recovers [20, Corollary 3.10] and [19, Theorem 2.6]. Next, in Theorem 4.12, we prove a complex version of 2.6 which improves [20, Theorem 3.8]. As a corollary, in 4.13, we deduce that $\text{Gid}_R \mathbf{H}_{\mathfrak{m}}^n(M) = \text{Gid}_R M - n$, wherever (R, \mathfrak{m}) is a local ring and M is a Cohen-Macaulay R -module with $\dim_R M = n$ (without assuming that R is Cohen-Macaulay). This corollary improves [20, Corollary 3.9].

2. RIGHT DERIVED SECTION FUNCTOR AND INJECTIVE DIMENSION

We recall the following lemma which shows that the right derived section functor can be computed via Čech complexes.

Lemma 2.1. *Let \mathfrak{a} be a proper ideal of R , and let M be an R -complex in $D(R)$. If $\underline{x} = x_1, \dots, x_r$ is a generating set for the ideal \mathfrak{a} and $\check{C}_{\underline{x}}$ the corresponding Čech complex, then*

$$\mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq M \otimes_R^{\mathbf{L}} \check{C}_{\underline{x}}.$$

Proof. see [15, Theorem 1.1(iv)]. \square

The following proposition is one of the main results of this section which has led to some interesting results.

Proposition 2.2. *Let (R, \mathfrak{m}, k) be a local ring, and let M be an R -complex in $D_{\square}(R)$. Then*

$$\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) \simeq \mathbf{R}\mathrm{Hom}_R(k, M).$$

Proof. Let F be a semifree resolution of the residue field k . Let $\underline{x} = x_1, \dots, x_r$ be a generating set for the ideal \mathfrak{a} and $\check{C}_{\underline{x}}$ be the Čech complex with respect to \underline{x} . Then, in view of [2, Theorem 4.4.5(a)], there are isomorphisms

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_R(k, M \otimes_R^{\mathbf{L}} \check{C}_{\underline{x}}) &\simeq \mathbf{R}\mathrm{Hom}_R(k, M) \otimes_R^{\mathbf{L}} \check{C}_{\underline{x}} \\ &\simeq \mathbf{R}\mathrm{Hom}_R(k, M) \otimes_k^{\mathbf{L}} (k \otimes_R^{\mathbf{L}} \check{C}_{\underline{x}}) \end{aligned}$$

in $D(R)$. But, as in the proof of [11, Lemma 2.4], there exists a quasiisomorphism $F \otimes_R \check{C}_{\underline{x}} \simeq F$ in $C(R)$. Hence $k \otimes_R^{\mathbf{L}} \check{C}_{\underline{x}} \simeq k$, and so

$$\mathbf{R}\mathrm{Hom}_R(k, M \otimes_R^{\mathbf{L}} \check{C}_{\underline{x}}) \simeq \mathbf{R}\mathrm{Hom}_R(k, M).$$

The result therefore follows from the fact that $M \otimes_R^{\mathbf{L}} \check{C}_{\underline{x}} \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(M)$. \square

Corollary 2.3. *Let (R, \mathfrak{m}, k) be a local ring, and let M be an R -complex in $D_{\square}(R)$. Then the following statements hold.*

- (i) $\mathrm{depth}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth}_R M$.
- (ii) *If R admits a dualizing complex D , then*

$$\mathrm{width}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)^{\dagger} = \mathrm{width}_R M^{\dagger}.$$

Proof. (i) Immediate from Proposition 2.2. (ii) By [2, Theorem 4.4.6(a)] and Proposition 2.2, there are isomorphisms

$$\begin{aligned} k \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M), D) &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\Gamma_{\mathfrak{a}}(M)), D) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, M), D) \\ &\simeq k \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(M, D) \end{aligned}$$

in $D(R)$. The result now follows by definition of width. \square

The following corollary, which is an immediate consequence of [10, Corollary 3.4.4] and 2.2, determines the i -th Bass number $\mu^i(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) := \mu_{R_{\mathfrak{p}}}^i(\mathbf{R}\Gamma_{\mathfrak{a}}(M)_{\mathfrak{p}})$ of $\mathbf{R}\Gamma_{\mathfrak{a}}(M)$, where is defined as in [2].

Corollary 2.4. *Let (R, \mathfrak{m}, k) be a local ring, and let M be an R -complex in $D_{\square}(R)$. Then $\mu_R^i(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \mu_R^i(M)$ for all $i \in \mathbb{Z}$; In particular, for every $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ there is an equality $\mu^i(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \mu^i(\mathfrak{p}, M)$ for all $i \in \mathbb{Z}$.*

Next, we recall the following lemma which is needed in the proof of Theorem 2.6.

Lemma 2.5. *Let M be a complex in $D(R)$. Then*

- (i) $\text{id}_R M = \sup\{-\inf \mathbf{R}\text{Hom}_R(T, M) \mid T \text{ is a cyclic } R\text{-module}\}$; and
- (ii) *If $M \in D_{\square}(R)$, then*

$$\text{id}_R M = \sup\{m \in \mathbb{Z} \mid \exists \mathfrak{p} \in \text{Spec} R : \mu_{R_{\mathfrak{p}}}^m(M_{\mathfrak{p}}) \neq 0\}.$$

Proof. Parts (i) and (ii) have been proved in [2, Theorem 5.1.6] and [2, Lemma 6.1.19], respectively. \square

The following theorem, which is one of the main results of this section, provides a comparison between the injective dimensions of a complex and its right derived section functor.

Theorem 2.6. *Let (R, \mathfrak{m}, k) be a local ring, and let M be an R -complex in $D_{\square}(R)$.*

- (i) *If $\text{id}_R M < \infty$, then $\text{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$.*
- (ii) *The converse holds whenever $M \in D_{\square}^f(R)$.*

Furthermore, if $M \in D_{\square}^f(R)$, then $\text{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{id}_R M$.

Proof. (i) Let $s := \text{id}_R M < \infty$. Then, in view of Lemma 2.5(ii) and Corollary 2.4, $\mu^{i+s}(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) = 0$ for all $\mathfrak{p} \in \text{Spec} R$ and for all $i > 0$. Therefore, it follows from Lemma 2.5(ii) that $\text{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq s$.

(ii) Let $t := \text{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Then, by Lemma 2.5(i), $\inf \mathbf{R}\text{Hom}_R(T, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) \geq -t$ for all cyclic R -module T . Hence, in view of Proposition 2.2, there are isomorphisms

$$\mathbf{H}_{-t-i}(\mathbf{R}\text{Hom}_R(k, M)) \cong \mathbf{H}_{-t-i}(\mathbf{R}\text{Hom}_R(k, \mathbf{R}\Gamma_{\mathfrak{a}}(M))) \cong 0$$

for all $i > 0$. Therefore $-\inf \mathbf{R}\text{Hom}_R(k, M) \leq t$, and so $\text{id}_R M \leq t < \infty$ by [2, Theorem 6.1.13]. The final assertion is a consequence of (i) and (ii). \square

The following corollary, which recovers [20, Corollary 2.7], is an immediate consequence of the previous Theorem.

Corollary 2.7. *Let (R, \mathfrak{m}) be a local ring. Then the following statements are equivalent:*

- (i) *R is Gorenstein;*
- (ii) *$\text{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) < \infty$ for any ideal \mathfrak{a} of R ;*
- (iii) *$\text{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) < \infty$ for some ideal \mathfrak{a} of R .*

Remark 2.8. Note that if M is an R -complex in $D(R)$ such that $\sup \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$, then there is an isomorphism $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ in $D(R)$. Hence $\Sigma^n \mathbf{R}\Gamma_{\mathfrak{a}}(M)$ is equivalent to the module $\Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ in the category of R -modules. We will abbreviate $\Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$ to $\mathbf{H}_{\mathfrak{a}}^n(M)$ when $\sup \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$.

Example 2.9. Let (R, \mathfrak{m}) be a Gorenstein local ring with $\dim(R) = d$, and let M be the R -complex

$$0 \longrightarrow \bigoplus_{\substack{\mathfrak{p} \in \text{Spec}(R) \\ \text{htp}=0}} E(R/\mathfrak{p}) \longrightarrow \bigoplus_{\substack{\mathfrak{p} \in \text{Spec}(R) \\ \text{htp}=1}} E(R/\mathfrak{p}) \longrightarrow \cdots \longrightarrow E(R/\mathfrak{m}) \longrightarrow 0.$$

Since $M \simeq R$, $\text{id}_R M = d$. On the other hand, $\mathbf{R}\Gamma_{\mathfrak{m}}(M) \simeq \mathbf{H}_{-d}(\mathbf{R}\Gamma_{\mathfrak{m}}(M))$, and so $\Sigma^d \mathbf{R}\Gamma_{\mathfrak{m}}(M) \cong E(R/\mathfrak{m})$. Therefore

$$\text{id}_R \mathbf{R}\Gamma_{\mathfrak{m}}(M) = d = \text{id}_R M.$$

The following theorem, which is an immediate consequence of Theorem 2.10, is a generalization of [20, Theorem 2.5].

Theorem 2.10. *Let (R, \mathfrak{m}) be a local ring, and let n be an integer. Suppose that M is an R -complex in $D_{\square}(R)$ such that $\sup \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$.*

- (i) *If $\mathrm{id}_R M < \infty$, then $\mathrm{id}_R \mathbf{H}_{\mathfrak{a}}^n(M) < \infty$.*
- (ii) *The converse holds whenever $M \in D_{\square}^f(R)$.*

Furthermore, if $M \in D_{\square}^f(R)$, then $\mathrm{id}_R \mathbf{H}_{\mathfrak{a}}^n(M) = \mathrm{id}_R M - n$.

Proof. Parts (i) and (ii) follow from Theorem 2.6 in view of Remark 2.8.

For the final assertion, we notice that, since $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$, there are equalities

$$\begin{aligned} \mathrm{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) &= \mathrm{id}_R \Sigma^{-n}(\Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))) \\ &= \mathrm{id}_R \Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) + n. \end{aligned}$$

Then, in view of Remark 2.8, $\mathrm{id}_R \mathbf{H}_{\mathfrak{a}}^n(M) = \mathrm{id}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) - n$. The desired equality now follows from Theorem 2.6. \square

3. RIGHT DERIVED SECTION FUNCTOR AND (GORENSTEIN) FLAT DIMENSION

The following proposition is one of the main results of this section which has led to some interesting results.

Proposition 3.1. *Let (R, \mathfrak{m}, k) be a local ring, and let M be an R -complex in $D(R)$. Then*

$$k \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq k \otimes_R^{\mathbf{L}} M.$$

Proof. Let $\underline{x} = x_1, \dots, x_r$ be a generating set for the ideal \mathfrak{a} and $\check{C}_{\underline{x}}$ be the Čech complex with respect to \underline{x} . Then, there exists an isomorphism $k \otimes_R^{\mathbf{L}} \check{C}_{\underline{x}} \simeq k$ in $D(R)$ (see 2.2). Hence, we have

$$k \otimes_R^{\mathbf{L}} \check{C}_{\underline{x}} \otimes_R^{\mathbf{L}} M \simeq k \otimes_R^{\mathbf{L}} M.$$

But $\check{C}_{\underline{x}} \otimes_R^{\mathbf{L}} M \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(M)$. Therefore $k \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq k \otimes_R^{\mathbf{L}} M$ as desired. \square

Corollary 3.2. *Let (R, \mathfrak{m}, k) be a local ring. Suppose that M and N are R -complexes in $D(R)$. Then the following statements hold.*

- (i) $\mathrm{width}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{width}_R M$.
- (ii) *If N is an R -complex in $D_{\square}(R)$ and $M \in D_{\square}(R)$, then*

$$\mathrm{depth}_R \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(M), N) = \mathrm{depth}_R \mathbf{R}\mathrm{Hom}_R(M, N).$$

In particular, If R admits a dualizing complex D , then

$$\mathrm{depth}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M)^{\dagger} = \mathrm{depth}_R M^{\dagger}.$$

Proof. (i) Immediate from Proposition 3.1. (ii) A straightforward application of [2, Proposition 5.2.6], part (i) and Corollary 2.3(i). \square

The following corollary, which is an immediate consequence of [10, Corollary 3.4.4] and 3.1, determines the i -th Betti number $\beta_i(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) := \beta_i^{R_{\mathfrak{p}}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)_{\mathfrak{p}})$ of $\mathbf{R}\Gamma_{\mathfrak{a}}(M)$, where is defined as in [2].

Corollary 3.3. *Let (R, \mathfrak{m}, k) be a local ring, and let M be an R -complex in $D(R)$. Then $\beta_i^R(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \beta_i^R(M)$ for all $i \in \mathbb{Z}$; In particular, for every $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ there is an equality $\beta_i(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) = \beta_i(\mathfrak{p}, M)$ for all $i \in \mathbb{Z}$.*

Next, we recall the following lemma which is needed in the proof of Theorem 3.5.

Lemma 3.4. *Let M be a complex in $D(R)$. Then*

- (i) $\text{fd}_R M = \sup\{\sup T \otimes_R^{\mathbf{L}} M \mid T \text{ is a cyclic } R\text{-module}\}$; and
- (ii) *There is an equality*

$$\text{fd}_R M = \sup\{m \in \mathbb{Z} \mid \exists \mathfrak{p} \in \text{Spec} R : \beta_m^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}.$$

Proof. Parts (i) and (ii) have been proved in [2, Theorem 5.1.9] and [2, Lemma 6.1.15], respectively. \square

Theorem 3.5, which is a flat version of 2.6, is one of the main results of this section. This theorem recovers some interesting results that have currently been appeared in the literature.

Theorem 3.5. *Let (R, \mathfrak{m}, k) be a local ring, and let \mathfrak{a} be a proper ideal of R . Suppose that M is an R -complex in $\mathbf{D}(R)$.*

- (i) *If $\text{fd}_R M < \infty$, then $\text{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$.*
- (ii) *The converse holds whenever $M \in \mathbf{D}_{\square}^f(R)$.*

Furthermore, if $M \in \mathbf{D}_{\square}^f(R)$, then $\text{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{fd}_R M$.

Proof. (i) Let $s := \text{fd}_R M < \infty$. Then, in view of Lemma 3.4(ii) and Corollary 3.3, $\beta_{i+s}(\mathfrak{p}, \mathbf{R}\Gamma_{\mathfrak{a}}(M)) = 0$ for all $\mathfrak{p} \in \text{Spec} R$ and for all $i > 0$. Therefore, it follows from Lemma 3.4(ii) that $\text{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq s$.

(ii) Let $t := \text{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Then, by Lemma 3.4(i), $\sup T \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{a}}(M) \geq t$ for all cyclic R -module T . Hence, in view of Proposition 3.1, there are isomorphisms

$$\mathbf{H}_{t+i}(k \otimes_R^{\mathbf{L}} M) \cong \mathbf{H}_{t+i}(k \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{a}}(M)) \cong 0$$

for all $i > 0$. Therefore $\sup k \otimes_R^{\mathbf{L}} M \leq t$, and so $\text{fd}_R M \leq t < \infty$ by [2, Theorem 6.1.13]. The final assertion is a consequence of (i) and (ii). \square

Example 3.6. Let (R, \mathfrak{m}) be a Gorenstein local ring with $\dim(R) = d$, and let M be as in 2.9. Since $M \simeq R$, $\text{fd}_R M = 0$. On the other hand, $\mathbf{R}\Gamma_{\mathfrak{m}}(M) \simeq \mathbf{H}_{-d}(\mathbf{R}\Gamma_{\mathfrak{m}}(M))$, and so $\Sigma^d \mathbf{R}\Gamma_{\mathfrak{m}}(M) \cong \mathbf{H}_{\mathfrak{m}}^n(R)$. Hence, by [21, Corollary 3.4], $\text{fd}_R \mathbf{R}\Gamma_{\mathfrak{m}}(M) + d = d$. Therefore

$$\text{fd}_R \mathbf{R}\Gamma_{\mathfrak{m}}(M) = 0 = \text{fd}_R M.$$

The following theorem, which is an immediate consequence of Theorem 3.5, is a flat version of 2.10.

Theorem 3.7. *Let (R, \mathfrak{m}) be a local ring, and let n be an integer. Suppose that M is an R -complex in $\mathbf{D}(R)$ such that $\sup \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$.*

- (i) *If $\text{fd}_R M < \infty$, then $\text{fd}_R \mathbf{H}_{\mathfrak{a}}^n(M) < \infty$.*
- (ii) *The converse holds whenever $M \in \mathbf{D}_{\square}^f(R)$.*

Furthermore, if $M \in \mathbf{D}_{\square}^f(R)$, then $\text{fd}_R \mathbf{H}_{\mathfrak{a}}^n(M) = \text{fd}_R M + n$.

Proof. Parts (i) and (ii) follow from Theorem 3.5 in view of Remark 2.8.

For the final assertion, we notice that, since $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \simeq \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))$, there are equalities

$$\begin{aligned} \text{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) &= \text{fd}_R \Sigma^{-n}(\Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))) \\ &= \text{fd}_R \Sigma^n \mathbf{H}_{-n}(\mathbf{R}\Gamma_{\mathfrak{a}}(M)) - n. \end{aligned}$$

Then, in view of Remark 2.8, $\text{fd}_R \mathbf{H}_{\mathfrak{a}}^n(M) = \text{fd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) + n$. The desired equality now follows from Theorem 3.5. \square

In the rest of this section, we make a comparison between the Gorenstein flat dimensions of a complex and its right derived section functor.

Proposition 3.8. *Suppose that M is an R -complex in $D_{\square}(R)$. Then*

$$\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \leq \mathrm{Gfd}_R M.$$

Proof. Notice that if $\mathrm{Gfd}_R M = \infty$, then there is nothing to prove. So, we may assume that $\mathrm{Gfd}_R M < \infty$. Hence, it follows from [4, Theorem 5.9] that $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Now, by [9, Theorem 8.8], there exists $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ such that

$$\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}.$$

But, by [10, Corollary 3.4.4] and Corollary 2.3(i), $\mathrm{depth}_{R_{\mathfrak{p}}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}} = \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. It follows, again by [9, Theorem 8.8], that

$$\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{depth} R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \mathrm{Gfd}_R M$$

as desired. \square

Proposition 3.9. *Let (R, \mathfrak{m}, k) be a local ring, and let M be an R -complex in $D_{\square}^f(R)$ such that $\mathrm{Gfd}_R M < \infty$. Then*

$$\mathrm{Gfd}_R M \leq \mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

Proof. By [4, Theorem 5.9], $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Hence, by [9, Theorem 8.7] and Corollary 2.3(i), there are equalities

$$\begin{aligned} \sup(E(k) \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{a}}(M)) &= \mathrm{depth} R - \mathrm{depth}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \\ &= \mathrm{depth} R - \mathrm{depth}_R M = \sup(E(k) \otimes_R^{\mathbf{L}} M). \end{aligned}$$

Since $M \in D_{\square}^f(R)$, $\sup(E(k) \otimes_R^{\mathbf{L}} M) = \mathrm{Gfd}_R M$. The result now follows from the fact that $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \geq \sup(E(k) \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{a}}(M))$ (see [4, Corollary 3.6]). \square

The following theorem, which is a Gorenstein flat version of Theorem 3.5, is one of the main results of this section.

Theorem 3.10. *Let (R, \mathfrak{m}) be a local ring, and let M be an R -complex in $D_{\square}^f(R)$.*

- (i) *If $\mathrm{Gfd}_R M < \infty$, then $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$.*
- (ii) *If R admits a dualizing complex, then $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \mathrm{Gfd}_R M$.*

Proof. (i) A straightforward application of Proposition 3.8 and Proposition 3.9.

(ii) In view of part (i), we may assume that $\mathrm{Gfd}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) < \infty$. Hence, by [4, Theorem 5.9], $\mathrm{Gfd}_R M < \infty$. The desired equality now follows from Proposition 3.8 and Proposition 3.9. \square

The next theorem is a Gorenstein flat version of Theorem 3.7.

Theorem 3.11. *Let (R, \mathfrak{m}) be a local ring admitting a dualizing complex, and let n be an integer. Suppose that M is an R -complex in $D_{\square}^f(R)$ such that $\sup \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$.*

- (i) *If $\mathrm{Gfd}_R M < \infty$, then $\mathrm{Gfd}_R \mathbf{H}_{\mathfrak{a}}^n(M) = \mathrm{Gfd}_R M + n$.*
- (ii) *If R admits a dualizing complex, then $\mathrm{Gfd}_R \mathbf{H}_{\mathfrak{a}}^n(M) = \mathrm{Gfd}_R M + n$.*

Proof. Follows from Theorem 3.10 in view of Remark 2.8. \square

Corollary 3.12. *Let (R, \mathfrak{m}) be a local ring. Suppose that M is relative Cohen-Macaulay with respect to \mathfrak{a} and that $n = \mathrm{grade}(\mathfrak{a}, M)$. Then $\mathrm{Gfd}_R \mathbf{H}_{\mathfrak{a}}^n(M) = \mathrm{Gfd}_R M + n$.*

Proof. Notice that $(\widehat{R}, \widehat{\mathfrak{m}})$ is a local ring admitting a dualizing complex and $M \otimes_R \widehat{R}$ is a relative Cohen-Macaulay \widehat{R} -module with respect to $\mathfrak{a}\widehat{R}$ and that $\mathrm{grade}(\mathfrak{a}\widehat{R}, M \otimes_R \widehat{R}) = n$. Hence, in view of [3, Theorem 4.27], we may assume that R is complete; and so it has a dualizing complex. The result therefore follows from Theorem 3.11. \square

4. RIGHT DERIVED SECTION FUNCTOR AND GORENSTEIN INJECTIVE DIMENSION

In this section the category of R -modules is denoted $\mathcal{C}(R)$. Recall from [1, Exercise 4.1.2] that the local cohomology modules of R -module M with respect to \mathfrak{a} can be calculated by an $\Gamma_{\mathfrak{a}}$ -acyclic resolution of M . First, we prove the complex version of this exercise.

Definition 4.1. (see [17]) Let $F : \mathcal{C}(R) \rightarrow \mathcal{C}(R)$ be a left exact functor, and assume that M is an R -complex in $C_{\square}(R)$. If $0 \rightarrow M \rightarrow C_{*,0} \rightarrow C_{*,1} \rightarrow \cdots \rightarrow C_{*,q} \rightarrow$ is a Cartan-Eilenberg injective resolutions of M , where is defined as in [13], define $\mathbb{R}^i(FM)$ to be $\mathbf{H}_i(\text{Tot}(FC))$.

Lemma 4.2. Let M and \acute{M} be two R -complexes in $C_{\square}(R)$, and let $\zeta : M \rightarrow \acute{M}$ be a morphism of R -complexes. Suppose that $0 \rightarrow M \rightarrow C_{*,0} \rightarrow C_{*,1} \rightarrow \cdots \rightarrow C_{*,q} \rightarrow$ and $0 \rightarrow \acute{M} \rightarrow \acute{C}_{*,0} \rightarrow \acute{C}_{*,1} \rightarrow \cdots \rightarrow \acute{C}_{*,q} \rightarrow$ are Cartan-Eilenberg injective resolutions of M and \acute{M} , respectively. Then there exists a sequence $\{\zeta_{*,q}\}_{q \in \mathbb{N}_0}$ of morphisms $\zeta_{*,q} : C_{*,q} \rightarrow \acute{C}_{*,q}$ of R -complexes over ζ .

Proof. A straightforward application of [12, Theorem 19]. \square

Lemma 4.3. Let $F : \mathcal{C}(R) \rightarrow \mathcal{C}(R)$ be a left exact functor, and let M and \acute{M} be two R -complexes. Then

- (i) Any quasiisomorphism $\zeta : M \rightarrow \acute{M}$ induces isomorphism

$$\mathbb{R}^i(FM) \cong \mathbb{R}^i(F\acute{M})$$

for all $i \in \mathbb{Z}$; and

- (ii) If M is F -acyclic R -complex in $C_{\square}(R)$, that is, M_i is F -acyclic for all $i \in \mathbb{Z}$, then

$$\mathbb{R}^i(FM) = \mathbf{H}_i(FM)$$

for all $i \in \mathbb{Z}$.

Proof. Straightforward verification similar to the proof of [17, Corollary 5.7.7]. \square

The following theorem, which is one of the main results of this section, provide us to prove some interesting results.

Theorem 4.4. Let $F : \mathcal{C}(R) \rightarrow \mathcal{C}(R)$ be a left exact functor, and let M be an F -acyclic R -complex in $C_{\square}(R)$. Assume that I is F -acyclic and $F(I)$ is injective for every injective R -module I . Then $F(M) \simeq F(E)$, for every semiinjective resolution $E \in C_{\square}(R)$ of M .

Proof. Let E be a semiinjective resolution of M with $E_v = 0$ for $v > \sup M$, and let $\zeta : M \xrightarrow{\simeq} E$ be an quasiisomorphism. By [13, Theorem 10.45], there exist Cartan-Eilenberg injective resolutions $0 \rightarrow M \rightarrow C_{*,0} \rightarrow C_{*,1} \rightarrow \cdots \rightarrow C_{*,q} \rightarrow$ and $0 \rightarrow E \rightarrow \acute{C}_{*,0} \rightarrow \acute{C}_{*,1} \rightarrow \cdots \rightarrow \acute{C}_{*,q} \rightarrow$. Hence, in view of Lemma 4.2, there is a sequence $\{\zeta_{*,q}\}_{q \in \mathbb{N}_0}$ of morphisms of R -complexes such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \longrightarrow & \acute{C}_{*,0} & \longrightarrow & \acute{C}_{*,1} & \longrightarrow & \cdots & \longrightarrow & \acute{C}_{*,q} & \longrightarrow \\ & & \uparrow \zeta & & \uparrow \zeta_{*,0} & & \uparrow \zeta_{*,1} & & & & \uparrow \zeta_{*,q} & \\ 0 & \longrightarrow & M & \longrightarrow & C_{*,0} & \longrightarrow & C_{*,1} & \longrightarrow & \cdots & \longrightarrow & C_{*,q} & \longrightarrow \end{array}$$

commutes in $C(R)$. By Lemma 4.3(ii), $\mathbb{R}^p(F(M)) = \mathbf{H}_p(F(M))$ for all $p \in \mathbb{Z}$, so that the natural morphism $F(M) \rightarrow \text{Tot}(F(C))$ is a quasiisomorphism. Similarly, $\mathbb{R}^p(F(E)) = \mathbf{H}_p(F(E))$ for all $p \in \mathbb{Z}$, so that the natural morphism $F(E) \rightarrow \text{Tot}(F(\acute{C}))$

is a quasiisomorphism. Thus, by [2, Proposition 3.3.5(a)], there exists a quasiisomorphism $\text{Tot}(F(\acute{C})) \rightarrow F(E)$, since $F(E)$ is injective.

But, by Lemma 4.3(i), there are isomorphisms $\mathbb{R}^p(F(M)) \cong \mathbb{R}^p(F(E))$ for all $p \in \mathbb{Z}$. Hence the morphism $\zeta_* : \text{Tot}(F(C)) \rightarrow \text{Tot}(F(\acute{C}))$ is a quasiisomorphism, where

$$\zeta_n = \sum_{p+q=n} F(\zeta_{p,q}) : \text{Tot}(F(C))_n \rightarrow \text{Tot}(F(\acute{C}))_n$$

for all $n \in \mathbb{Z}$. Therefore, there are quasiisomorphisms

$$F(M) \xrightarrow{\sim} \text{Tot}(F(C)) \xrightarrow{\sim} \text{Tot}(F(\acute{C})) \xrightarrow{\sim} F(E).$$

Now $F(M) \simeq F(E)$ as desired. \square

The next theorem, which offers an application of the previous theorem, is a complex version of [1, Exercise 4.1.2].

Theorem 4.5. *Let C be an Γ_a -acyclic R -complex in $C_{\square}(R)$. Then $\mathbf{R}\Gamma_a(C) \simeq \Gamma_a(C)$.*

It should be noted that the following corollary, as an application of Theorem 4.5, follows from the result of J. Lipman in [10, Corollary 3.4.3]; however, the proof below is different from his.

Corollary 4.6. *Let $f : R \rightarrow R'$ be a ring homomorphism of commutative Noetherian rings, and let $[_R] : \mathcal{C}(R') \rightarrow \mathcal{C}(R)$ to denote the functor obtained from restriction of scalars (using f). Suppose that M' is an R' -complex in $D_{\square}(R')$. Then there is an isomorphism*

$$\mathbf{R}\Gamma_a(M'[_R]) \simeq \mathbf{R}\Gamma_{aR'}(M')[_R]$$

in $D(R)$. In particular, $\mathbf{H}_{-i}(\mathbf{R}\Gamma_a(M'[_R])) \cong \mathbf{H}_{-i}(\mathbf{R}\Gamma_{aR'}(M'))[_R]$ for all $i \in \mathbb{Z}$.

Proof. Let $I' \in C_{\square}(R')$ be an injective resolutions of M' . Note that $M'[_R] \simeq I'[_R]$. But, by [1, Theorem 4.1.6], $I'[_R]$ is Γ_a -acyclic R -complexes in $C_{\square}(R)$, and so, by Theorem 4.5, there is an isomorphism

$$\mathbf{R}\Gamma_a(M'[_R]) \simeq \Gamma_a(I'[_R])$$

in $D(R)$. The result now follows, since $\Gamma_a(I'[_R]) = \Gamma_{aR'}(I')[_R]$. \square

It has been proved in [14, Corollary 3.3] that if R admits a dualizing complex and $M \in D_{\square}(R)$, then $\text{Gid}_R \mathbf{R}\Gamma_a(M) \leq \text{Gid}_R M$. The next corollary together with [14, Theorem 3.2] recover this result. Also, notice that [14, Theorem 3.4] is an immediate consequence of 4.7.

Corollary 4.7. *Let M be an R -complex in $D_{\square}(R)$, and let $G \in C_{\square}(R)$ be an R -complex of Gorenstein injective modules such that $M \simeq G$. Then $\mathbf{R}\Gamma_a(M) \simeq \Gamma_a(G)$.*

Proof. Since by [18, Lemma 1.1] every Gorenstein injective module is Γ_a -acyclic, the result follows from Theorem 4.5. \square

The following proposition together with [4, Theorem 5.9] recover [14, Corollary 3.3].

Proposition 4.8. *Suppose that M is an R -complex in $D_{\square}(R)$ such that $\text{Gid}_R \mathbf{R}\Gamma_a(M) < \infty$. Then*

$$\text{Gid}_R \mathbf{R}\Gamma_a(M) \leq \text{Gid}_R M.$$

Proof. Notice that if $\text{Gid}_R M = \infty$, then there is nothing to prove. So, we may assume that $\text{Gid}_R M < \infty$. By [5, Theorem 2.2], there exists $\mathfrak{p} \in \mathbf{V}(\mathfrak{a})$ such that

$$\text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{depth}_{R_{\mathfrak{p}}} - \text{width}_{R_{\mathfrak{p}}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}}.$$

But, by [10, Corollary 3.4.4] and Corollary 3.2(i), $\text{width}_{R_{\mathfrak{p}}}(\mathbf{R}\Gamma_{\mathfrak{a}}(M))_{\mathfrak{p}} = \text{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$. It follows, again by [5, Theorem 2.2], that

$$\text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{depth}_{R_{\mathfrak{p}}} - \text{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \text{Gid}_R M$$

as desired. \square

Proposition 4.9. *Let (R, \mathfrak{m}) be a local ring, and let M be an R -complex in $D_{\square}^f(R)$ such that $\text{Gid}_R M < \infty$. Then*

$$\text{Gid}_R M \leq \text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

Proof. By [4, Proposition 6.3], there is an inequality

$$\text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) \geq \text{depth}_R - \text{width}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M).$$

But, by Corollary 3.2(i), $\text{width}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{width}_R M$. Thus, we have

$$\begin{aligned} \text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) &\geq \text{depth}_R - \text{width}_R M \\ &= \text{depth}_R - \inf M. \end{aligned}$$

The result therefore follows from [5, Corollary 2.3]. \square

The following theorem, which is a Gorenstein injective version of Theorem 2.6, is one of the main results of this section.

Theorem 4.10. *Let (R, \mathfrak{m}) be a local ring admitting a dualizing complex, and let M be an R -complex in $D_{\square}^f(R)$. Then $\text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \text{Gid}_R M$.*

Proof. A straightforward application of [4, Theorem 5.9], Proposition 4.8 and Proposition 4.9. \square

In [20, Corollary 3.10] a characterization of a Gorenstein local ring R is given. The next corollary recovers that characterization. Also, notice that Corollary 4.11 together with Corollary 2.7 recover [19, Theorem 2.6].

Corollary 4.11. *Let (R, \mathfrak{m}) be a local ring admitting a dualizing complex. Then the following statements are equivalent:*

- (i) R is Gorenstein;
- (ii) $\text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) < \infty$ for any ideal \mathfrak{a} of R ;
- (iii) $\text{Gid}_R \mathbf{R}\Gamma_{\mathfrak{a}}(R) < \infty$ for some ideal \mathfrak{a} of R .

Proof. A straightforward application of Theorem 4.10 and [3, Proposition 3.11]. \square

The next theorem, which is a Gorenstein injective version of Theorem 2.10, recovers [20, Theorem 3.8].

Theorem 4.12. *Let (R, \mathfrak{m}) be a local ring admitting a dualizing complex, and let n be an integer. Suppose that M is an R -complex in $D_{\square}^f(R)$ such that $\sup \mathbf{R}\Gamma_{\mathfrak{a}}(M) = \inf \mathbf{R}\Gamma_{\mathfrak{a}}(M) = -n$. Then $\text{Gid}_R \mathbf{H}_{\mathfrak{a}}^n(M) = \text{Gid}_R M - n$.*

Proof. Follows from Theorem 4.10 in view of Remark 2.8. \square

Let (R, \mathfrak{m}) be a local ring, and let M be an R -module. As a result, it has been shown in [20, Corollary 3.9] that if R and M are Cohen-Macaulay with $\dim_R M = n$, then $\text{Gid}_R \mathbf{H}_{\mathfrak{m}}^n(M) = \text{Gid}_R M - n$. We show that [20, Corollary 3.9] holds without assuming that R is Cohen-Macaulay.

Corollary 4.13. *Let (R, \mathfrak{m}) be a local ring, and let M be a Cohen-Macaulay R -module with $\dim_R M = n$. Then the following statements hold.*

- (i) $\text{Gid}_{\widehat{R}} \mathbf{R}\Gamma_{\widehat{\mathfrak{a}}\widehat{R}}(M \otimes_R \widehat{R}) = \text{Gid}_R M$.
- (ii) $\text{Gid}_R \mathbf{H}_{\mathfrak{m}}^n(M) = \text{Gid}_R M - n$.

Proof. Notice that $(\widehat{R}, \widehat{\mathfrak{m}})$ is a local ring admitting a dualizing complex and $M \otimes_R \widehat{R}$ is a Cohen-Macaulay \widehat{R} -module of dimension n .

- (i) A straightforward application of Theorem 4.10 and [3, Theorem 3.24].
- (ii) It follows, by Remark 2.8, that

$$\text{Gid}_{\widehat{R}} \mathbf{R}\Gamma_{\widehat{\mathfrak{m}}}(M \otimes_R \widehat{R}) = \text{Gid}_{\widehat{R}} \mathbf{H}_{\widehat{\mathfrak{m}}}^n(M \otimes_R \widehat{R}) + n$$

But, in view of [14, Lemma 3.6], $\text{Gid}_{\widehat{R}} \mathbf{H}_{\widehat{\mathfrak{m}}}^n(M \otimes_R \widehat{R}) = \text{Gid}_R \mathbf{H}_{\mathfrak{m}}^n(M)$. The result now follows from part (i). \square

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C. JALALI, FACULTY OF MATHEMATICAL SCIENCES AND COMPUTER, KHARAZMI UNIVERSITY, 599
TALEGHANI AVENUE, TEHRAN 15618, IRAN

E-mail address: c.jalali@yahoo.com